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Swimming in Space-Time

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Abstract

Cyclic changes in the shape of a quasi-rigid body on a curved manifold can lead to net translation and/or rotation of the body in the manifold. Presuming space-time is a curved manifold as portrayed by general relativity, translation in space can be accomplished simply by cyclic changes in the shape of a body, without any thrust or external forces.

1 Introduction

The motion of a swimmer at low Reynolds number is determined by the geometry of the sequence of shapes that the swimmer assumes (Shapere and Wilczek, 1989a). At low Reynolds number the effects of inertia are negligible, and in the absence of external forces bodies are at rest. Nevertheless, as a body changes its shape its location and orientation generally change. A cyclic change in the shape of a body can lead to a net translation or rotation. The net translation or rotation does not depend on the speed with which the shape changes are carried out; it is a consequence of the geometry of the sequence of shapes. It is an example of geometric phase.[1] It has been proposed that the cilia of a paramecium effectively define and allow changes in its shape, and that locomotion is accomplished through cyclic changes in its shape through motion of its cilia.

Consider two concentric spheres with equal moments of inertia connected at their centers (Shapere and Wilczek, 1989b, hereafter SW1989). Suppose the spheres are initially not rotating. Now rotate one sphere with respect to the other sphere. The orientation of the system of two spheres adjusts appropriately. The adjustment can be determined using the fact that angular momentum is conserved. If the total angular momentum is zero, the angular velocities of the spheres, with equal moments of inertia, are equal but opposite. So if one sphere is rotated about an axis by an angle θ with respect to the other sphere then the resulting motion of the system is that the first sphere rotates in space about that axis by $\theta/2$ and the other sphere rotates by $-\theta/2$. If the two spheres undergo a sequence of rotations with respect to each other and are then brought back to their original relative orientation then their orientation in space can undergo a net rotation. The net rotation of the system does not depend on the speed with which these relative rotations occur; it depends only on the sequence of relative orientations. It is another example of geometric phase.

Generalizing the latter example, it is shown here that quasi-rigid bodies can swim on curved manifolds simply by cyclic changes in their shape. Presuming space-time is a curved manifold as portrayed by general relativity, net translations in space can be accomplished through cyclic, engineered changes in the shape of a body. Motion in space can be accomplished without thrust or external forces.

[1] Geometric phase is also called anholonomy.

2 Dynamics of Connected Rigid Bodies

Consider a system consisting of two bodies that have a common fixed point. The bodies can change their relative configuration by rotating with respect to each other. The problem is to determine the consequent change in orientation of this compound body in space as it changes its internal configuration in a specified way.

The configuration of the system is specified by giving the relative orientation of the two bodies and the orientation in space of the compound body. The orientation of a body can be specified by giving the rotation that takes the body from some starting position to the orientation being specified. For this compound body this is done in two stages. First, with both bodies initially in a starting position, the bodies are rotated to give a specifed relative configuration. Each relative configuration has a standard reference orientation. Then from this reference orientation for that relative configuration the compound body is rotated to its actual orientation. Each body has a starting orientation, a reference orientation for each relative configuration, and an actual orientation.

The reference configuration of the system at some particular time is specified by starting each constituent body in a starting orientation and then specifying the rotation that takes the constituent from its starting configuration to its orientation in the reference configuration. It is convenient to choose the starting orientations of the constituents so that the principal axes of the constituent are aligned with an inertial right-handed orthonormal basis $(\hat{x}, \hat{y}, \hat{z})$. The inertia tensors of the constituent bodies in this starting orientation, I^s_i for body i, are diagonal. A convenient choice for the reference configurations of the system is to hold one body, labelled α, fixed and rotate the other body, labelled β, relative to it. Then the orientation of body α is the same as the orientation of the system. The reference orientation of body α is the same as its starting orientation, so the inertia tensor of body α in its reference orientation is $I'_\alpha(t) = I^s_\alpha$. Body β starts with its principal axes aligned with the inertial basis (and with the principal axes of body α) and is brought to its reference orientation at time t by an active rotation $M_c(t)$. The inertia tensor of body β in the starting orientation, I^s_β, is diagonal; the inertia tensor of body β in its reference orientation, $I'_\beta(t) = M_c(t) I^s_\beta (M_c(t))^{-1}$, is not, in general, diagonal. In the reference orientation the inertial basis has a particular orientation with respect to the system. An orthonormal set of body axes can be defined by rotating these axes, which are aligned with the inertial axes in the reference configuration, by the same active rotation that rotates the body. These body axes move as the system moves. If the configuration of the body did not change these axes could, for instance, be taken to be the body-fixed principal axes. Here, the principal axes and principal moments change with time, and it is not required that the body axes bear any particular relationship to the instantaneous principal axes. In general, primes indicate components that are taken with respect to the body axes. A superscript s indicates components in the starting orientation, which may be the same as the reference orientation (as for body α).

The orientation of the system at time t is specified by the active rotation $M(t)$ that takes the compound system in its reference orientation to the actual orientation at time t. This in turn might be specified in terms of a rotation parametrized by a tuple $q(t)$ of three generalized coordinates that depend on time: $M(t) = R(q(t))$. At time t the rotation $M_\alpha(t) = M(t)$ rotates body α from its starting orientation, which is its reference position, to the actual orientation, and the rotation $M_\beta(t) = M(t)M_c(t)$ rotates body β from its starting orientation to its reference orientation and then to its actual orientation. The function M_c is given as part of the specification of the system; the function M is determined by the dynamical evolution of the system.

The motion of the system is governed by the Lagrange equations. As there is no potential energy, the kinetic energy is a Lagrangian for the system. The kinetic energy of the system is the sum of the kinetic energy of the constituents. The kinetic energy of body i is

$$\frac{1}{2}(A_i(\omega_i^a)^2 + B_i(\omega_i^b)^2 + C_i(\omega_i^c)^2), \tag{1}$$

where $(\omega_i^a, \omega_i^b, \omega_i^c)$ are the components of the angular velocity vector of body i on the principal axes, and (A_i, B_i, C_i) are the principal moments of inertia of body i, with respect to the common center about which the two bodies rotate. The tuple of components of the angular velocity vector of body i are (see Sussman and Wisdom, 2001, hereafter SW2001):

$$\omega_i(t) = \mathsf{A}^{-1}(DM_i(t)(M_i(t))^{-1}), \tag{2}$$

where A is the linear function that maps the tuple of components (x, y, z) of a vector $\vec{u}$ to the skew-symmetric matrix that represents the cross product operator $\vec{u}\times$:

$$\mathsf{A}(x, y, z) = \left[\begin{pmatrix} 0 \\ x \\ -y \end{pmatrix} \quad \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \quad \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} \right] \tag{3}$$

The function A^{-1} is the functional inverse of A, and $(M_i(t))^{-1}$ is the multiplicative inverse of $M_i(t)$. The multiplicative inverse of $M_i(t)$ can be represented by the transpose of the matrix that represents $M_i(t)$. The derivative operator is D, the derivative of the function M is DM, and $DM(t)$ is the value of the derivative of M at time t. The tuple of components of the angular velocity of the system at time t is

$$\omega(t) = \mathsf{A}^{-1}(DM(t)(M(t))^{-1}), \tag{4}$$

and

$$\omega'_c(t) = \mathsf{A}^{-1}(DM_c(t)(M_c(t))^{-1}) \tag{5}$$

is the tuple of components of an angular velocity associated with the configuration change. The angular velocities of the constituent bodies are

$$\begin{aligned} \omega_\alpha(t) &= \omega(t) & (6)\\ \omega_\beta(t) &= \mathsf{A}^{-1}(D(MM_c)(t))(M(t)M_c(t))^{-1}) \\ &= \omega(t) + \omega_c(t), & (7) \end{aligned}$$

where $\omega_c(t) = M(t)\omega'_c(t)$. Equation (6) says that the angular velocity of body α is the same as the angular velocity of the system; Equation (7) says that the angular velocity of body β is that of the system plus an angular velocity associated with the deformation, rotated to the current orientation. The components of the angular velocities on the principal axes of the appropriate body can be obtained by rotating the angular velocity tuples back to the starting orientations (for which the principal axes are aligned with the fixed inertial axes and the inertia tensor is diagonal):

$$\begin{aligned} \omega^s_\alpha(t) &= (M(t))^{-1}\omega_\alpha(t) & (8)\\ \omega^s_\beta(t) &= (M(t)M_c(t))^{-1}\omega_\beta(t) \\ &= (M_c(t))^{-1}(M(t))^{-1}\omega_\alpha(t) + (M_c(t))^{-1}\omega'_c(t). & (9) \end{aligned}$$

Having obtained the components of the angular velocities on the principal axes, the kinetic energy of the system is given by equation (1). To finish the specification of the Lagrangian function a set of generalized coordinates $q(t)$ must be chosen that parametrize the rotation: $M(t) = R(q(t))$. For instance, the Euler angle parametrization could be chosen. There is, however, no need to show this explicitly. The Lagrange equations, which govern the evolution of the system, will also not be shown explicitly.

The derivation of the Lagrangian used an arbitrary fixed reference orthonormal basis $(\hat{x}, \hat{y}, \hat{z})$; any alternate set obtained from these by an arbitrary rotation would do equally as well. In other words, the Lagrangian has rotational invariance. Noether's theorem associates continuous symmetries to conserved momenta. In this case the vector angular momentum, which is the sum of the vector angular momenta of the constituents, is conserved. The components of the angular momentum with respect to any fixed reference basis are all conserved. The tuple of components of the angular momentum on the principal axes of body i are $L_i^s(t) = I_i^s \omega_i^s(t)$, with I_i^s diagonal. The tuple of components with respect to the fixed inertial basis are $L_i(t) = M_i(t) L_i^s(t)$. The tuple of components of the total angular momentum on the body axes is

$$L'(t) = L_\alpha^s(t) + M_c(t) L_\beta^s(t). \tag{10}$$

The tuple of spatial components of the total angular momentum is

$$L(t) = L_\alpha(t) + L_\beta(t) = M(t) L'(t). \tag{11}$$

The total angular momentum is conserved, so $L(t)$ is constant: $DL(t) = 0$. Rewriting this in terms of the L'

$$0 = DM(t)L'(t) + M(t)DL'(t), \tag{12}$$

so

$$\begin{aligned} DL'(t) &= -(M(t))^{-1} DM(t) L'(t) \\ &= -(M(t))^{-1} \mathsf{A}(\omega(t)) M(t) L'(t) \\ &= -\mathsf{A}(\omega'(t)) L'(t), \end{aligned} \tag{13}$$

where $\omega'(t) = (M(t))^{-1}\omega(t)$ is the tuple of components of the angular velocity on the body axes. These are Euler's equations. These equations express the fact that the angular momentum vector is fixed in space, but the components with respect to changing basis vectors tied to the system change in a way that depends on the angular velocity of the system.

The angular momentum can be written as the sum of the angular momentum associated with the instantaneous configuration and the angular momentum associated with the change of the configuration:

$$L'(t) = I'(t)\omega'(t) + h'(t), \tag{14}$$

where $\omega'(t)$ is the tuple of components of the angular velocity with respect to the body fixed basis, and the inertia tensor of the system with respect to the body fixed basis is

$$\begin{aligned} I'(t) &= I'_\alpha(t) + I'_\beta(t) \\ &= I_\alpha^s + M_c(t) I_\beta^s (M_c(t))^{-1}. \end{aligned} \tag{15}$$

The inertia tensor is time dependent because the configuration of the system depends on time. The expression for $h'(t)$ can be deduced by comparing Equations (14) and (10):

$$h'(t) = M_c(t) I_\beta^s (M_c(t))^{-1} \omega_c'(t) = I_\beta'(t) \omega_c'(t). \tag{16}$$

This, $h'(t)$, is the tuple of components of the angular momentum of deformation on the body basis at time t. The function h' does not depend on the dynamical state of the system; it is determined solely by the time-dependent deformation. In terms of L' and h' the tuple of components of the angular velocity of the system on the body basis is $\omega'(t) = (I'(t))^{-1}(L'(t) - h'(t))$. Euler's equations can then be rewritten

$$DL'(t) = -\mathsf{A}((I'(t))^{-1}(L'(t) - h'(t)))L'(t). \tag{17}$$

These are Liouville's equations for the motion of a deformable body. Note that $I'(t)$ is not necessarily diagonal.

Collecting the relevant equations, the evolution of the system is governed by:

$$DL'(t) = -\mathsf{A}(\omega'(t))L'(t), \tag{18}$$

with

$$\omega'(t) = (I'(t))^{-1}(L'(t) - h'(t)) \tag{19}$$

and

$$DM(t) = \mathsf{A}(M(t)\omega'(t))M(t), \tag{20}$$

or

$$DM(t) = M(t)\mathsf{A}(\omega'(t)). \tag{21}$$

The conserved angular momentum is

$$\begin{aligned} L(t) &= I(t)\omega(t) + I_\beta(t) M(t) \omega_c'(t) \\ &= I(t)\omega(t) + I_\beta(t)\omega_c(t). \end{aligned} \tag{22}$$

For states for which $L = 0$ the angular velocity of the system is

$$\omega(t) = -(I(t))^{-1} I_\beta(t) \omega_c(t), \tag{23}$$

or

$$\omega'(t) = -(I'(t))^{-1} I_\beta'(t) \omega_c'(t). \tag{24}$$

This equation relates the body components of the system angular velocity to the body components of the angular velocity associated with the deformation. The inertia tensors are not necessarily diagonal.

Suppose the bodies are spheres, with I_α and I_β both diagonal with all moments of inertia equal to C. The system moments are then $2C$ at any time (about any axis). So in this special case

$$\omega(t) = -\frac{1}{2}\omega_c(t), \tag{25}$$

or

$$\omega'(t) = -\frac{1}{2}\omega_c'(t). \tag{26}$$

At any moment the system rotates in the opposite direction to the rotation of deformation at that time at half the rate.

Now how do changes in orientation accumulate? Given $\omega'(t)$, equation (21) is a linear differential equation that can be integrated, for appropriate initial data, to give $M(t)$. Alternatively, one may express $M(t)$ in terms of generalized coordinates $R(q(t))$ and integrate the corresponding differential equations for the coordinate path q.

The solution to equation (21) may be represented formally as a path-ordered integral:

$$\begin{aligned} M(t) &= M(t_0)\bar{P}\left(\exp\left(\int_{t_0}^{t} \mathsf{A}(\omega'(t'))dt'\right)\right) \\ &= M(t_0)\left(I + \int_{t_0<t'<t} \mathsf{A}(\omega'(t'))dt'\right. \\ &\quad \left. + \int_{t_0<t'<t''<t} \mathsf{A}(\omega'(t'))\mathsf{A}(\omega'(t''))dt'dt'' + \cdots\right), \end{aligned} \tag{27}$$

where I is the identity multiplier compatible with $M(t)$. The second expression defines the reverse path-ordered integral that is indicated by $\bar{P}$ in the first expression. All factors in the integrands of the multiple integrals are ordered with larger times to the right. The rotation $M_c(t)$ can be similarly expressed as a path-ordered integral, with $M_c(t)$ replacing $M(t)$ and $\omega_c'(t)$ replacing $\omega'(t)$.

Alternatively, equation (4) has the formal solution

$$\begin{aligned} M(t) &= P\left(\exp\left(\int_{t_0}^{t} \mathsf{A}(\omega(t'))dt'\right)\right)M(t_0) \\ &= \left(I + \int_{t>t'>t_0} \mathsf{A}(\omega(t'))dt'\right. \\ &\quad \left. + \int_{t>t''>t'>t_0} \mathsf{A}(\omega(t''))\mathsf{A}(\omega(t'))dt'dt'' + \cdots\right) M(t_0), \end{aligned} \tag{28}$$

where the second line defines the path-ordered integral that is indicated by P in the first line. All factors in the integrands are written with larger times to the left. And there is a similar representation for $M_c(t)$.

The factor $\mathsf{A}(\omega'(t))$ can be written in alternate ways that are useful. Recall that the function A is linear. So

$$\mathsf{A}(\omega'(t)) = \omega'(t)J = \sum_i (\omega'(t))^i J_i, \tag{29}$$

where J_i are the generators of infinitesimal rotations about the ith coordinate axis: $J_0 = \mathsf{A}(1,0,0)$, $J_1 = \mathsf{A}(0,1,0)$, and $J_2 = \mathsf{A}(0,0,1)$. Alternatively, because A is linear, $J = D\mathsf{A}(0,0,0)$, where J is the tuple of infinitesimal generators (see SW2001). The generators satisfy the commutation relations $[J_0, J_1] = J_2$, plus cyclic permutations. The rotation $R_i(\epsilon) = \exp(\epsilon J_i)$ actively rotates vectors to which it is applied about the ith basis vector by the angle ϵ.

Sequences of simple rotations about particular axes can be composed to generate particular $M_c(t)$ that represent simple deformations. During each simple rotation the angular velocity vector $\omega'_c(t) = w'_c$ is constant, and it acts during a specified time interval with length τ. The rotation has the exponential representation $\exp(\tau w'_c J)$. The change in $M_c(t)$ during this interval is

$$M_c(t+\tau) = M_c(t)\mathrm{e}^{\tau w'_c J}. \tag{30}$$

The path-ordered integral reduces to this expression for constant $\omega'_c(t)$ because the factors $\mathsf{A}(\omega'_c(t))$ are constant and equal and so can be factored out of the integrals, which then give the coefficients of the exponential series. Using equation (24) the system angular velocity $\omega'(t)$ that corresponds to $\omega'_c(t)$ can be determined. In general, even if $\omega'_c(t)$ is constant, $\omega'(t)$ is not constant, because the inertia tensors are time-dependent.

However, in the special case in which both bodies are spheres (or any body for which all principal moments are equal), the inertia tensors are independent of orientation and constant, so if $\omega'_c(t) = w'_c$ is constant then $\omega'(t) = w'$ is also constant. A sequence of uniform deformation rotations then leads to a corresponding sequence of uniform rotations of the system. In this case $w' = -\gamma w'_c$, where γ is ratio of the moment of inertia of sphere β to the total moment of inertia of the system. If the spheres have equal moments of inertia then $\gamma = 1/2$. A sequence of deformation rotations leads to

$$\begin{aligned} M_c(t_f) &= \mathrm{e}^{\epsilon J_0}\mathrm{e}^{\epsilon J_1}\mathrm{e}^{-\epsilon J_0}\mathrm{e}^{-\epsilon J_1}\mathrm{e}^{-\epsilon^2[J_0,J_1]} \\ &= I + o(\epsilon^3) \end{aligned} \tag{31}$$

where ϵ is the angle of rotation (the product of the time interval and the magnitude of the angular velocity) for the first four rotations, and t_f is the time at which these rotations are completed. Note that the sequence of rotations are represented by successive factors from left to right, consistent with equations (27) and (30). The corresponding rotation of the system is

$$\begin{aligned} M(t) &= \mathrm{e}^{-\gamma\epsilon J_0}\mathrm{e}^{-\gamma\epsilon J_1}\mathrm{e}^{\gamma\epsilon J_0}\mathrm{e}^{\gamma\epsilon J_1}\mathrm{e}^{\gamma\epsilon^2[J_0,J_1]} \\ &= \mathrm{e}^{(\gamma-\gamma^2)\epsilon^2 J_2} + o(\epsilon^3). \end{aligned} \tag{32}$$

Each system rotation has an extra factor of $-\gamma$ in the exponent, compared to the corresponding deformation rotation. Even though the system undergoes a sequence of deformations that brings it back to its original configuration to order ϵ^3, the system undergoes a net rotation of order ϵ^2 (SW1989). The configuration could be brought back exactly to its starting configuration with an additional rotation of order ϵ^3, and the net rotation would remain of order ϵ^2.

An interesting feature that is demonstrated by this example is that the net rotation of the system depends on the sequence of rotations and the magnitude of the angles of rotation, but not on how fast the deformation takes place. The net rotation depends only on the path of deformation. The net rotation is a geometric phase (SW1989).

Think about this for a minute. Two spheres with zero total angular momentum can perform a sequence of relative rotations involving angles of order ϵ and undergo a net rotation in space of order ϵ^2. If this sequence is repeated regularly, then an average net rotation per unit time ensues. Doing this sequence faster gives rise to a faster net rate of rotation. Now ϵ can be made as small as one likes, and the system will have the same net rate of rotation if the sequence is repeatedly carried out appropriately rapidly. Indeed, ϵ can be made so small that the relative motion of the two spheres is microscopic, making the relative rotations essentially unobservable, yet the system has a net rate of rotation. And, all the while, the angular momentum of the system is zero.

The system has a gauge symmetry (SW1989). The choice of reference orientation for each relative configuration was chosen arbitrarily: the principal axes of body α were aligned with the inertial basis. Other choices are possible (see below). The rotation that takes the system from the reference orientation to the actual orientation is $M(t) = R(q(t))$, and R is a function of generalized coordinates that give this rotation for generalized coordinates $q(t)$ along the coordinate path q. A new reference orientation can be chosen for each configuration by applying a rotation that depends on the configuration. So if $\Omega(q(t))$ is the extra rotation that takes the system from the old reference position (for generalized coordinates $q(t)$) to the new reference position, then the rotation that takes the system from its new reference orientation to the actual orientation is

$$\widetilde{M}(t) = R(q(t))(\Omega(q(t)))^{-1} = M(t)(\Omega(q(t)))^{-1}. \tag{33}$$

The first rotation takes the system back to the old reference orientation, and then $M(t) = R(q(t))$ rotates it to the actual orientation. The equations of motion, equations (18) to (21), all must have the same form, since the choice of reference orientations is arbitrary. In particular, equation (21) must be

$$D\widetilde{M}(t) = \widetilde{M}(t)\mathsf{A}(\widetilde{\omega}'(t)). \tag{34}$$

The requirement that this is equivalent to the equation of motion in the original gauge determines $\mathsf{A}(\widetilde{\omega}'(t))$:

$$\mathsf{A}(\widetilde{\omega}'(t)) = \Omega(q(t))\mathsf{A}(\omega'(t))(\Omega(q(t)))^{-1} - \Omega(q(t))\frac{d}{dt}((\Omega(q(t)))^{-1}), \tag{35}$$

which is the rule for the transformation of a gauge potential.

3 Generalization

Though the discussion has focused on the particular example of two concentric spheres, most of the equations that have been derived apply to any two bodies with a common fixed point. The bodies do not have to be spheres. The equations apply equally well to two bodies of arbitrary shape with a rigid constraint to a common fixed point. They also apply to the limiting case of rigid spherical caps of arbitrary shape constrained to move on the surface of a sphere. A cyclic change in the relative orientation of two spheres can lead to a net rotation of the system. In the case of two spherical caps, a cylic change in the relative positions of the spherical caps on the sphere can lead to a net motion of the system around the sphere. In more picturesque language, a two-dimensional bug can crawl on the surface of a frictionless sphere just by cyclic changes in its shape.

Consider two circular spherical caps of angular radius γ on a sphere of radius R. Let m be the mass of each cap, with uniform surface density. The moment of inertia of each cap about the radial line through its center is

$$C = 2mR^2\left(\sin\frac{\gamma}{2}\right)^2\left(1 - \frac{2}{3}\left(\sin\frac{\gamma}{2}\right)^2\right), \tag{36}$$

which for small γ is approximately $m\gamma^2 R^2/2$. The moment of inertia about any line through the center of the sphere perpendicular to this radial line is

$$A = mR^2 \left(1 - \left(\sin\frac{\gamma}{2}\right)^2 + \frac{2}{3}\left(\sin\frac{\gamma}{2}\right)^4\right), \tag{37}$$

which for small γ is approximately mR^2. The relative configuration is specified by two angles. A symmetrical reference configuration is convenient, specified as follows (see figure 1). Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be a right-handed orthonormal basis centered at the center of the sphere. For the starting configuration place both bodies with their centers on the $\hat{x}$ axis. The reference configuration of body α is obtained by two successive active rotations: a rotation by θ about the $\hat{x}$ axis, followed by a rotation by ϕ about the $\hat{y}$ axis. The reference configuration of body β is obtained by the opposite rotations: a rotation by $-\theta$ about the $\hat{x}$ axis, followed by a rotation by $-\phi$ about the $\hat{y}$ axis. This symmetrically places the centers of the two caps above and below the equator and rotates them oppositely. The configuration of the system is changed by changing the two configuration parameters θ and ϕ. Because of the symmetry of the system it will move by rotating around the $\hat{z}$ axis. Let ψ be this dynamical degree of freedom. Assuming zero total angular momentum, the equation of motion of ψ is

$$D\psi(t) = \frac{D\theta(t) C \sin\phi(t)}{A(\cos\phi(t))^2 + C(\sin\phi(t))^2}. \tag{38}$$

The denominator is half the moment of inertia of the system about the $\hat{z}$ axis; the numerator is half the $\hat{z}$ component of the angular momentum due to the twisting (non-zero $D\theta$) of the cap. Each cap makes the same contribution to the $\hat{z}$ moment of inertia and the $\hat{z}$ component of the angular momentum, so the additional factors of two in the numerator and denominator cancel. Note that the counter rotation of the two disks can be accomplished without external torques. This can be accomplished, for example, by fixing the distance between the centers by a rigid circular arc, and then accomplishing the counter rotation by contracting a tension wire symmetrically attached to the outer edge of the two caps. Nevertheless, their contributions to the $\hat{z}$ component of the angular momentum are parallel and add to give a non-zero angular momentum. Other components of the angular momentum are zero. Because of the fact that the angular momentum of the system is zero, the angular momentum due to twisting must be balanced by the angular momentum of the motion of the system.

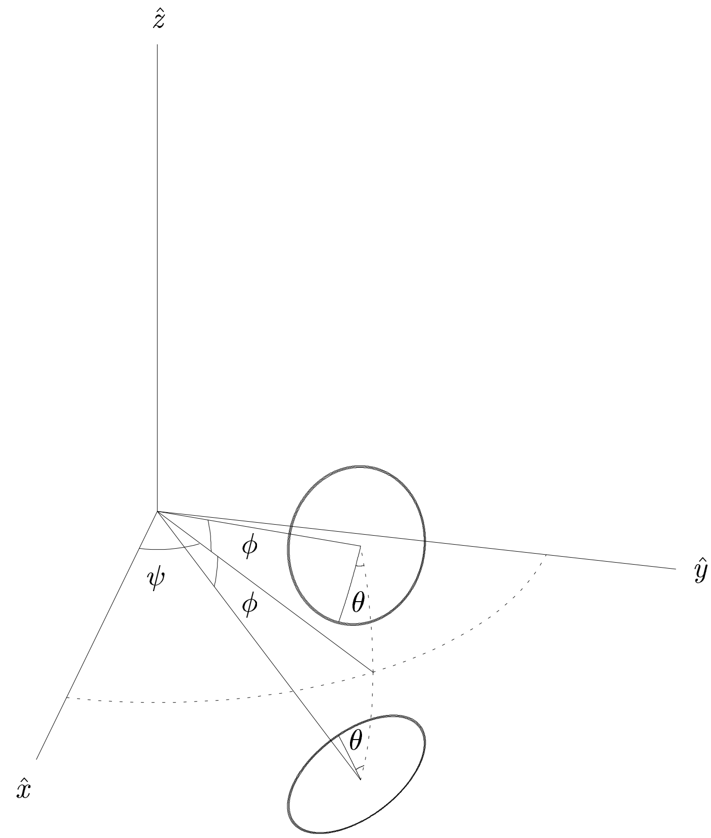

Figure 1: Two spherical caps on a sphere. The two deformation parameters are θ and ϕ. The caps twist oppositely by the angle θ, and are symmetrically displaced by the angle ϕ above and below the equator. The angle ψ is the longitude of the system. The deformation parameters follow a specified schedule, the longitude of the system adjusts dynamically.

A net rotation of the system can be accomplished by taking the internal configuration of the system through a cycle. The angles θ and ϕ may be considered deformation coordinates. Contrast the fact that here the deformation is specified by coordinates (the associated basis vector fields commute) with the fact that the successive angles of rotation used in the two-sphere example are not coordinates (the associated vector fields do not commute). A simple cycle may be accomplished by increasing θ by $\Delta\theta$ holding ϕ fixed, then increasing ϕ by $\Delta\phi$ holding θ fixed, decreasing θ, then decreasing ϕ, returning to the original relative configuration. In those phases of this cycle in which θ is held fixed the system does not rotate. The rotation of the system during the two phases in which θ is changed do not balance because ϕ is different.

The rotation of the system during a cycle is a geometric phase, it does not depend on the speed at which the cycle is traversed. In this case, the total rotation of the system can be written as a line integral

$$\Delta\psi = \oint_{\partial\Pi} \omega, \tag{39}$$

around the boundary $\partial\Pi$ of the region Π in the plane of deformation parameters (θ, ϕ) of the one-form ω

$$\omega(\theta, \phi) = \frac{C \sin\phi}{A(\cos\phi)^2 + C(\sin\phi)^2} d\theta. \tag{40}$$

Using Stokes's theorem, the change in ψ can equally well be written as an integral over the region Π in deformation parameters

$$\Delta\psi = \int_{\Pi} d\omega. \tag{41}$$

The one-form ω is not closed, so $\Delta\psi$ is generally not zero, and for small deformations is roughly proportional to the area of the region enclosed by the path in the plane of deformation parameters. For small γ (small caps), small ϕ (small separation), and small $\Delta\theta$ and $\Delta\phi$ (small deformations), the net motion of the system per cycle of deformation is approximately

$$\Delta\psi = \frac{1}{2}\gamma^2 \, \Delta\theta \, \Delta\phi. \tag{42}$$

Having made everything small compared to the size of the sphere, it is apparent that the essence of the matter is not that these are rigid bodies, but that the system lives on a curved manifold.

4 Swimming on Curved Manifolds

It is also possible to swim on manifolds of non-constant curvature, but the idea of rigid bodies must first be generalized. Rigid bodies, defined by a large number of redundant distance constraints between constituents, cannot, in general, move on a manifold of non-constant curvature because the redundant constraints cannot remain consistent. Consider three masses on the vertices of a triangle, with geodesics of fixed length connecting them. Now place a fourth mass in the middle of the triangle, equidistant from each of the three masses. The distance to those masses will depend on the curvature of the manifold. This system of four masses cannot move to a region of different curvature without some of the constraint distances changing. This is a consequence of the fact that the constraints are redundant—more constraints are specified than are needed to specify the relative location of the masses. If instead of specifying that the central mass is equidistant from the three masses it is specified that it is a certain distance from two of the vertices then all of the constraints can be maintained even as the curvature changes. Such a rigid body with irredundant constraints will be called a quasi-rigid body. A quasi-rigid body has a well defined configuration even though the distances between all the constituents are not fixed. One particular class of quasi-rigid bodies has a tree-like topology. From each vertex extend an arbitrary number of branches or struts. At the end of any strut, there may be attached more struts. Masses may be attached to the vertices. The configuration of the system is determined if the lengths of all the struts and the angles they make with the connecting struts are specified. The first example shows that more general quasi-rigid bodies are allowed, they need not be tree-like. And the struts need not be geodesics. Struts of quasi-rigid bodies can be constructed from a local lattice or truss of nearly rigid rods. If the rods are small compared to the intrinsic curvature of the manifold then they can maintain the local constraints of the truss, without much strain, while conforming to the manifold on a larger scale. Think of metal strapping material laid out on a putting green.

For a quasi-rigid body to swim on a curved manifold it must undergo changes in its shape. For such a change to result in a net rotation or translation of the body on the manifold the shape changes must go through a non-trivial cycle of deformation, so the dimension of the space of deformations must be at least two-dimensional. A simple quasi-rigid body that satisfies these requirements is a body consisting of one mass point with mass m_0 connected to two other mass points with mass m_1 by geodesic struts of given length separated by a given angle. The body can be deformed by changing the length of the struts or the angle between them.

Consider the dynamics of such a quasi-rigid body on two illustrative manifolds: the plane and the sphere, embedded in flat three-dimensional space. It is informative to consider these manifolds even though they have constant curvature, and so the assumption of quasi-rigidity is not required. Assume the motion of free particles in the embedding space is described by the usual non-relativistic kinetic energy Lagrangian $\frac{1}{2}mv^2$. A Lagrangian for the body can be obtained by defining coordinates that incorporate the constraints of both the body and the manifold, and writing the free-particle Lagrangian in terms of these coordinates (see SW2001). The motion of the system can be found by solving the expression for the conserved momenta for the rates of change of the dynamical variables in terms of the deformation parameters and their rates of change. In this case, where the momenta are linear in the generalized velocities, this involves the solution of linear equations. By symmetry (of the body and the manifold), the simple three-mass quasi-rigid body will move only along the direction bisecting the two struts. The calculation can be simplified by choosing one of the manifold coordinates to coincide with this direction.

For the plane, choose rectangular coordinates (x, y), with the x-axis bisecting the struts. The coordinates of the vertex at time t are $(x(t), y(t))$. The length of the strut is l and the angle between the struts and the x-axis is α. Momentum conservation, with zero momentum, leads to

$$\begin{aligned} Dx(t) &= \frac{2m_1}{m_0 + 2m_1}\left(l(t)\sin\alpha(t)D\alpha(t) - \cos\alpha(t)Dl(t)\right) \\ Dy(t) &= 0. \end{aligned} \tag{43}$$

For a cycle of deformation in the parameter plane (l, α), the net translation is

$$\Delta x = \oint_{\partial\Pi} \omega, \tag{44}$$

where $\partial\Pi$ is the parameter path, the boundary of a region Π in the parameter plane, and the integral is over the one-form

$$\omega(l, \alpha) = \frac{2m_1}{m_0 + 2m_1}\left(l \sin\alpha d\alpha - \cos\alpha dl\right). \tag{45}$$

This one-form is closed, so there is no net translation for a cycle of deformation: $\Delta x = 0$.

For the sphere, choose spherical coordinates with latitude θ and longitude ψ, with the equator bisecting the struts. The coordinates of the vertex at time t are $(\theta(t), \psi(t))$. The length of the geodesic struts, which are spherical

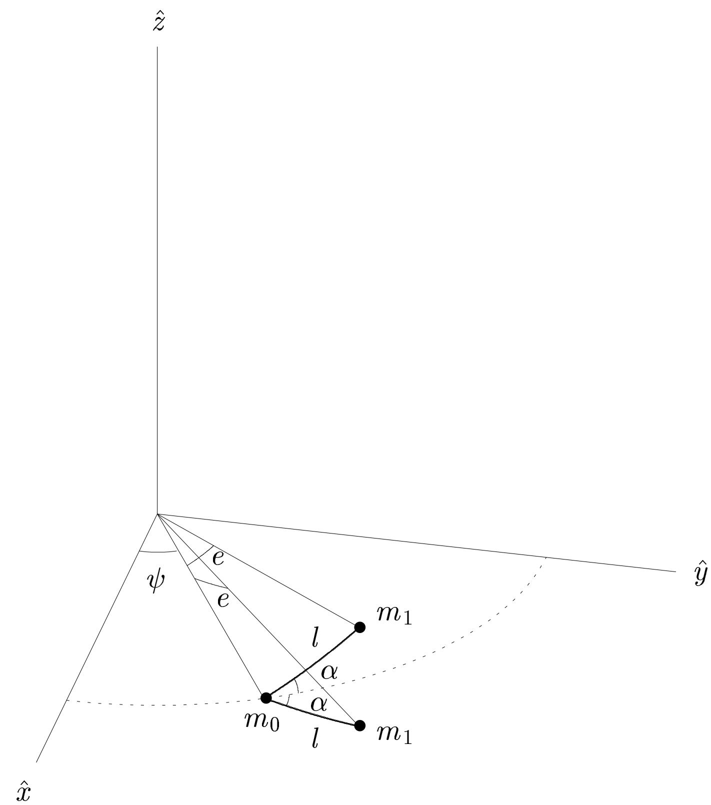

Figure 2: A quasi-rigid body on the sphere. The two deformation parameters are the length l and separation angle α. The geodesic struts subtend the angle $e = l/R$ at the center of the sphere, where R is the radius of the sphere, and the separation angle α is the angle between the struts and the symmetry line, measured on the plane tangent to the sphere at the vertex. The angle ψ is the longitude of the system. The deformation parameters follow a specified schedule, the longitude of the system adjusts dynamically.

arcs, is specified by the angular extent e, measured from the center of the sphere. The separation angle between the struts and the line of symmetry is α (see figure 2). (The angle between the struts is 2α.) Conservation of momentum, with zero initial momentum, is enough to determine the motion of the system as it deforms. Conservation of the momentum conjugate to θ gives a relation between $D\theta$ and $D\psi$. Using this and the conserved momentum conjugate to ψ leads to

$$\begin{aligned} D\psi(t) &= \frac{2m_1 \sin e(t) \cos e(t) \sin \alpha(t)}{m_0 + 2m_1(\cos\alpha(t))^2 + 2m_1(\sin\alpha(t))^2(\cos e(t))^2} D\alpha(t) \\ &\quad - \frac{2m_1 \cos\alpha(t)}{m_0 + 2m_1(\cos e(t))^2 + 2m_1(\cos\alpha(t))^2(\sin e(t))^2} De(t). \end{aligned} \tag{46}$$

For a cycle of deformation in the parameter plane (e, α), the net translation is

$$\Delta\psi = \oint_{\partial\Pi} \omega, \tag{47}$$

where $\partial\Pi$ is the parameter path and the integral is over the one-form

$$\begin{aligned} \omega(e,\alpha) &= \frac{2m_1 \sin e \cos e \sin\alpha}{m_0 + 2m_1(\cos\alpha)^2 + 2m_1(\sin\alpha)^2(\cos e)^2} d\alpha \\ &\quad - \frac{2m_1 \cos\alpha}{m_0 + 2m_1(\cos e)^2 + 2m_1(\cos\alpha)^2(\sin e)^2} de. \end{aligned} \tag{48}$$

This one-form is not closed, so there is a net translation, non-zero $\Delta\psi$, for a cycle of deformation in the (e, α) plane. Using Stokes's theorem the net rotation can be written as an integral of $d\omega$ over the region Π in the deformation parameter plane. For a small rectangular deformation region, with edges parallel to the deformation coordinates of size Δe and $\Delta\alpha$, the net translation is approximately

$$\Delta\psi = d\omega(\partial/\partial e, \partial/\partial\alpha)(e, s)\Delta e \Delta\alpha, \tag{49}$$

where $\partial/\partial e$ and $\partial/\partial\alpha$ are the coordinate basis vectors in the deformation plane. The two-form $d\omega$, given the two basis vector arguments, is a real valued function on the deformation parameter plane (e, α). For bodies with small extent relative to the size of the sphere, this is approximately

$$\Delta\psi = -\frac{4m_0 m_1}{(m_0 + 2m_1)^2} (\sin e)^2 \, \sin\alpha \, \Delta e \Delta\alpha. \tag{50}$$

The plane is flat; motion in the plane cannot be accomplished by internal deformations. The surface of the sphere has intrinsic curvature, and extrinsic curvature with respect to the three-dimensional embedding space; motion on the sphere using only internal deformation is possible.

5 Swimming on Manifolds without Symmetry

For both the plane and the sphere, a symmetrical swimmer swims in one direction without twisting. The amount of motion per stroke can written as a line integral of a real-valued one-form over the parameter path. Using Stokes's theorem this can also be written as the integral of a real-valued two-form over the parameter region enclosed by the parameter path. For systems without symmetry, the motion is given as an integral of a vector-valued one-form over the parameter path. For small parameter loops around parallelograms defined by two parameter vectors, the net motion is a vector-valued two-form on the two vectors defining the parameter region. The vector-valued one-form can be viewed as a vector gauge potential, and the associated vector-valued two-form is the field strength. The expressions reduce to those in the symmetric case, for which the motion is one-dimensional.

The equations governing the motion of an idealized quasi-rigid deformable swimmer made up of a number of constituent masses with time-dependent (controlled) positional constraints among them are derived as follows. Let $L_\alpha(t, x, \dot{x})$ be the Lagrangian governing the motion of unconstrained particles of mass m_α at time t for the tuple of generalized manifold coordinates x, and the associated tuple of generalized velocities $\dot{x}$. Generally the free Lagrangians do not depend on the formal time argument. The time-dependent constraints are introduced as a coordinate transformation that gives the coordinates of the constituents in terms of the generalized coordinates q of the system (see SW2001). The coordinates of constituent α are functions of the tuple of deformation parameters $c(t)$ at time t, which vary in time in a specified way, and the generalized coordinates q

$$x_\alpha = f_\alpha(c(t), q), \tag{51}$$

and the associated velocities are[2]

$$\dot{x}_\alpha = \partial_0 f_\alpha(c(t), q) Dc(t) + \partial_1 f_\alpha(t, q)\dot{q}, \tag{52}$$

where $\dot{q}$ are the generalized velocities associated with q. The Lagrangian for the system is obtained by composition (see SW2001)

$$L(t, q, \dot{q}) = \sum_\alpha L_\alpha(t, f_\alpha(c(t), q), \partial_0 f_\alpha(c(t), q) Dc(t) + \partial_1 f_\alpha(c(t), q)\dot{q}). \tag{53}$$

[2]The notation is that of SW2001. Derivatives with respect to structured arguments have a dual structure so that if multiplied by an increment in those arguments contracts to approximate the increment in the function.

The momenta conjugate to $\dot{q}$ are

$$\begin{aligned} P(t, q, \dot{q}) &= \partial_2 L(t, q, \dot{q}) \\ &= \sum_\alpha \partial_2 L_\alpha(t, x_\alpha, \dot{x}_\alpha) \partial_1 f_\alpha(t, q), \end{aligned} \tag{54}$$

where the arguments of $\partial_2 L_\alpha$ are to be written in terms of q and $\dot{q}$ using equations (51) and (52).

The examples so far have relied on the fact that a momentum is conserved, and have made use this conserved momentum to deduce the inertial motion of the system from changes in the deformation parameters. If the momentum is not conserved, this does not work. However, if the parameters are varied quickly, then the response of the system is almost the same as if the momenta were conserved. Assume this is the case and consider henceforth the case for which the momentum is zero, whether or not it is exactly conserved or only approximately conserved for fast deformations. Generalization to non-zero conserved momenta (approximately or exactly) is straightforward.

For Lagrangians that are homogeneous quadratic forms in the generalized velocities the momenta are linear in the velocites. For this case the mass matrix

$$M_\alpha(t, x_\alpha) = \partial_2 \partial_2 L_\alpha(t, x_\alpha, \bullet) \tag{55}$$

is independent of velocity, and the system momentum is

$$P(t, q, \dot{q}) = \sum_\alpha (M_\alpha(t, f_\alpha(c(t), q))(\partial_0 f_\alpha(c(t), q) Dc(t) + \partial_1 f_\alpha(t, q)\dot{q})) \partial_1 f_\alpha(t, q). \tag{56}$$

For constant momentum, this is a set of linear equations that can be solved to give $\dot{q}$ in terms of t, q, $c(t)$, and $Dc(t)$. The result is linear in $Dc(t)$. On a solution path q the result is

$$Dq(t) = A(c(t), q(t)) Dc(t). \tag{57}$$

An example was given above for motion in the plane, equations (43). This may be viewed as defining a vector valued one-form $\mathcal{A}$

$$\mathcal{A}(c, q) = A(c, q) dc, \tag{58}$$

which takes tangent vectors along the deformation parameter path, with components $Dc(t)$, to the generalized velocity components.

A specific path through the deformation parameter space can be chosen by giving the tangent as a function of time $Dc(t) = \eta(t)$. The evolution in the state space $s = (t, c, q)$ is governed by the equations

$$\begin{aligned} Dt(\tau) &= 1 \\ Dc(\tau) &= \eta(\tau) \\ Dq(\tau) &= A(c(\tau), q(\tau))\eta(\tau), \end{aligned} \tag{59}$$

or

$$Ds(\tau) = G_{\eta(\tau)}(s(\tau)). \tag{60}$$

With appropriate initial conditions, $t(\tau) = \tau$. Let

$$L_G F = DF\, G, \tag{61}$$

be the operator that gives the rate of change of state functions F along solution paths of G. Exponentiating this operator advances the state (see SW2001)

$$s(t_0 + \Delta t) = (\mathrm{e}^{\Delta t\, L_G} I)(s(t_0)). \tag{62}$$

Advancing the coordinate selector gives the coordinates

$$q(t_0 + \Delta t) = (\mathrm{e}^{\Delta t\, L_G} Q)(t_0, c(t_0), q(t_0)), \tag{63}$$

where $Q(t, c, q) = q$.

Consider the evolution of the system resulting from a small loop in deformation parameter space around a parallelogram specified by two vectors ξ_0 and ξ_1. Along each segment $\eta(t) = \xi_i$. Let G_{ξ_i} be G specialized to this constant deformation parameter vector. The loop is traversed by moving first along ξ_0, then ξ_1, then $-\xi_0$, returning to the initial point with $-\xi_1$. The coordinate tuple after evolution around the parallelogram is

$$q(t_0 + \Delta t) = (\mathrm{e}^{L_{G_{\xi_0}}} \mathrm{e}^{L_{G_{\xi_1}}} \mathrm{e}^{-L_{G_{\xi_0}}} \mathrm{e}^{-L_{G_{\xi_1}}} Q)(t_0, c(t_0), q(t_0)). \tag{64}$$

For small loops the lowest order change in the coordinates is given by the commutator

$$q(t_0 + \Delta t) = q(t_0) + ([L_{G_{\xi_0}}, L_{G_{\xi_1}}]Q)(t_0, c(t_0), q(t_0)). \tag{65}$$

The commutator defines a vector valued two-form of the vectors ξ_0 and ξ_1. This two-form is the field strength $\mathcal{F}$ associated with the gauge vector potential $\mathcal{A}$. Let dc^a and dc^b be dual basis one-forms on the parameter plane. The components of the vector potential are $A^i(c, q) = A^i_a(c, q)dc^a + A^i_b(c, q)dc^b$

with i running through the component selectors of $\dot{q}$. The components of the field strength are[3]

$$
\begin{aligned}
F^i(\xi_0, \xi_1)(c, q) &= ([L_{G_{\xi_0}}, L_{G_{\xi_1}}]Q)(\bullet, c, q) \\
&= ((\partial_1 A^i_b(c, q))A_a(c, q) - (\partial_1 A^i_a(c, q))A_b(c, q))(\xi^a_0 \xi^b_1 - \xi^b_0 \xi^a_1) \\
&\quad + ((\partial_0 B^i(\xi_1))(c, q))\xi_0 - ((\partial_0 B^i(\xi_0))(c, q))\xi_1, \qquad (66)
\end{aligned}
$$

where

$$
B^i(\xi)(c, q) = A^i \xi = A^i_a(c, q)\xi^a + A^i_b(c, q)\xi^b. \qquad (67)
$$

The field strength can be used to determine whether cyclic deformation results in translation for more complicated geometries than so far considered, whether or not symmetries are present. A non-zero field strength F^i implies there is a change in coordinate q^i as the parameters traverse the loop specified by the two deformation parameter vectors ξ_0 and ξ_1. In the cases so far considered, where the translation was along a single coordinate, the corresponding component of the field strength reduces to the real-valued two-form specified earlier.

6 Swimming with Intrinsic Curvature

Swimming on a frictionless plane cannot be accomplished by cyclic deformation of shape but swimming on a sphere works. The plane has no intrinsic or extrinsic curvature. The sphere has both. It is therefore interesting to consider whether swimming on a cylinder is possible. The cylinder has extrinsic curvature but has no intrinsic curvature. The symmetries of the cylinder are ignored to illustrate the general method. The details follow.

Assume the radius of the cylinder is R. Use the usual cylindrical coordinates $q = (\theta, z)$, where θ measures the angle around the clyinder and z the height. Construct a body from three point masses. The angle ϕ will specify the orientation of the body. The cylindrical coordinates of the vertex of the body, with mass m_0, are (θ, z). Two other point masses, with mass m_1, are placed a length $l(t)$ away from the first body. The angle between the each strut and the bisector is $\alpha(t)$. The bisector of the two struts is tilted by the angle ϕ with respect to the horizonal (which is perpendicular to the axis of the cylinder). Assume the free Lagrangian for each mass is

[3] In this expression, keep in mind that the structure produced by the ∂_1 contracts with the vector components of the following A. Similarly, the structure generated by the ∂_0 contracts with the components of the following ξ. See SW2001.

$\frac{1}{2}mv^2$ in the three-dimensional embedding space. The Lagrangian of the system is obtained by composing the free Lagrangian with the time dependent constraints, extended to states (see SW2001).

After some calculation, the components of the gauge potential are as follows:

$$
\begin{aligned}
A_l(l,\alpha;\theta,z,\phi) &= \begin{pmatrix} \frac{-2m_1\cos\phi\cos\alpha}{R(m_0+2m_1)} \\ \\ \frac{-2m_1\sin\phi\cos\alpha}{R(m_0+2m_1)} \\ \\ 0 \end{pmatrix} \\
A_\alpha(l,\alpha;\theta,z,\phi) &= \begin{pmatrix} \frac{2lm_1\cos\phi\sin\alpha}{R(m_0+2m_1)} \\ \\ \frac{2lm_1\sin\phi\sin\alpha}{(m_0+2m_1)} \\ \\ 0 \end{pmatrix}
\end{aligned} \tag{68}
$$

The fact that the ϕ component of the gauge potentials is zero means the tilt of the body does not change as the body deforms. The other components are non-zero, so there is some motion on the cylinder. However, computation of the field strength shows it to be identically zero

$$F(\xi_0,\xi_1)(l,\alpha;\theta,z,\phi) = (0,0,0), \tag{69}$$

for arbitrary deformation vectors. So small cyclic deformations of this body on a cylinder do not give a net translation. Apparently, intrinsic curvature is required for swimming on curved manifolds, and extrinsic curvature does not help.

7 Swimming in Curved Space-Time

As shown above, it is possible for a quasi-rigid body to swim on a manifold with intrinsic curvature through cyclic deformations of shape. General relativity portrays space-time as a curved four-dimensional manifold. One immediately wonders whether it is possible to swim in space-time through cyclic deformations.

The idea of a quasi-rigid body overcomes the difficulty of maintaining constraints as a body moves from place to place where the curvature of the manifold changes, by removing redundancy in the positional constraints. In relativity there is another difficulty. Forces of constraint move with finite velocity, so if one part of the system receives an impulse there will be a delay in the response of other parts of the system. Natural bodies are not described by pure positional constraints. However, there is no obstacle to engineering a quasi-rigid body that does maintain positional constraints, as long as the schedule of deformations of the body are known sufficiently in advance. In this case the internal stresses that are required to maintain the positional constraints can be precomputed and prespecified, and executed simultaneously. The engineered quasi-rigid body is choreographed for a particular frame, which defines simultaneity. Ballet is not Lorentz invariant. It is choreographed so that dancers make simultaneous movements in the frame of the audience. In other frames, the dancers would be out of sync, but those observers are invited to slow down and enjoy the performance. This idea of engineered quasi-rigid bodies is introduced so that the analysis follows the previous examples; it seems likely that it would not be strictly required to be able to swim in space-time.

Of most interest is the curved space-time around a non-rotating mass, described by Schwarzschild geometry. The Schwarzschild coordinates are $q = (t, r, \theta, \phi)$ and $\xi_i = (\xi_i^t, \xi_i^r, \xi_i^\theta, \xi_i^\phi)$ are components of tangent vectors, with respect to the Schwarzschild coordinate basis vectors. The Schwarzschild metric g is (see, e.g., Misner, Thorne, and Wheeler, 1970)

$$\begin{aligned} g(q)(\xi_0, \xi_1) \\ = \ -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \xi_0^t \xi_1^t + \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)} \xi_0^r \xi_1^r + r^2 (\xi_0^\theta \xi_1^\theta + (\sin\theta)^2 \xi_0^\phi \xi_1^\phi), \end{aligned} \tag{70}$$

where M is the mass, G is the gravitational constant, and c is the velocity of light.

According to general relativity test particles follow geodesics on this curved space-time manifold. Geodesics are solutions q of the Lagrange equations for the free Lagrangian

$$L(\tau, q, u) = \frac{1}{2} g(q)(u, u), \tag{71}$$

using the Schwarzschild metric g. The formal parameters of the Lagrangian are the affine parameter along the path, the generalized coordinates, here the Schwarzschild coordinates, and the generalized velocities associated with the coordinates, here the four-velocity. Light rays are also geodesics of the same Lagrangian. Test particles and light rays are distinguished by their initial conditions. The magnitude of the four-velocity of a test particle is c, while for a light ray it is zero. The fact that the trajectories are solutions of the Lagrange equations implies the trajectories are trajectories of stationary action with respect to variations of the path q that hold the coordinates at the endpoints fixed.

This Lagrangian is a homogeneous quadratic form in the generalized velocities. The Legendre transform is non-singular, so there is an equivalent Hamiltonian description of the dynamics. The fact that the Lagrangian has no explicit τ dependence implies the Hamiltonian also has no explicit τ dependence. This in turn implies that the Hamiltonian is a conserved quantity, the value is constant along solution trajectories. In fact, the value is the same for all test particle trajectories—it is just $-c^2/2$. This integral can be used to eliminate one degree of freedom. It is convenient to make t be the new independent variable of the reduced phase space and eliminate τ. Normally such a reduction can only be done on an energy surface (for given values of the Hamiltonian), but here the Hamiltonian has the same value for all particle paths, so the reduction is general. There is, however, no need to show the details. By making Schwarzschild time the independent variable, the motion of many particles can be followed with this common time parametrizing them all. This will allow the dance of the swimmer to be choreographed in a frame of constant Schwarzschild time.

To construct a swimmer (a quasi-rigid deformable body) time-dependent positional constraints are introduced. It is more natural to introduce constraints in the Lagrangian framework. The reduced Hamiltonian has an equivalent Lagrangian description (Bertschinger, 1999). The reduced Lagrangian for a particle of mass m is

$$
\begin{aligned}
&L_3(t; r, \theta, \phi; \dot{r}, \dot{\theta}, \dot{\phi}) \\
&\quad = \quad mc \left(c^2 \left(1 - \frac{2GM}{c^2 r}\right) - \left(\frac{\dot{r}^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2 (\dot{\theta}^2 + (\sin\theta)^2 \dot{\phi}^2) \right) \right)^{1/2} \qquad (72)
\end{aligned}
$$

The independent variable is Schwarzschild time. The coordinates are the Schwarzschild spatial coordinates, and the generalized velocities are the rates of change of the Schwarzschild coordinates with respect to Schwarzschild time.

The construction of the active struts requires some discussion. The struts are constructed at fixed Schwarzschild time, and have a specified proper length between the endpoints. Following the earlier examples the struts are taken to be geodesics on the space-time manifold at fixed time. The struts follow solutions of the Lagrange equations with the Lagrangian

$$L_s(\lambda; x; v) = \frac{1}{2} g(q)(u, u), \qquad (73)$$

where g is again the Schwarzschild metric, the space-time coordinates are $q = (t, x)$, where t is the Schwarzschild time, and $u = (0, v)$. The components of the generalized velocity v are the rates of change of the Schwarzschild spatial coordinates x with respect to the independent variable λ, which is proportional to the proper length along the strut. A network of such active struts can maintain any desired positional constraints among the masses of a quasi-rigid body.

Keep in mind that the struts must be constantly monitored to make sure that the constraints are maintained. This monitoring may be done locally along the strut by carefully surveying neighboring points on the strut. The monitoring may be done arbitrarily quickly by giving each surveyor responsibility for an arbitrarily small segment. Also keep in mind that each surveyor must know in advance what stresses to apply so that the system maintains the required largescale positional constraints.

Consider the following space-time swimmer (see figure 3). Place one point mass, with mass m_0, at the vertex. Then extend three equal length struts, with proper length $l(t)$. In a local Lorentz frame, let $\alpha(t)$ be the angle each of these struts makes with an axis defining the orientation of the body, and distribute the struts equally in angle about the axis. For three struts, the angle between them is $2\pi/3$ radians. (If the axis of the body were aligned with the north pole, then α would be the colatitude of the struts, and the longitudinal angle between the struts would be $120°$.) At the end of each of these three struts place a point mass, with mass m_1. The deformation parameters are $l(t)$ and $\alpha(t)$. The system Lagrangian is obtained from the individual free Lagrangians as before, by writing the free variables in terms of the system variables that incorporate the constraints. The assumption is that the constrained system follows trajectories of stationary action.

The calculation is simplified by introducing one additional assumption. The free Lagrangian L_3 is not quadratic in the velocities, so the mass matrix depends on the generalized velocities. In this case, the solution for the generalized velocities in terms of the rates of change of the deformation parameters would involve the solution of nonlinear equations. But in the limit where the velocities are small compared to the velocity of light, the mass matrix is constant. For simplicity, this assumption is made here.

The goal is to determine whether swimming in space-time is possible. So it is enough to consider a special orientation of the body. If the axis of the body is oriented radially away from the central mass, then the symmetry of the Schwarzschild geometry and the three-fold symmetry of the swimmer guarantee that any translation due to cyclic deformation will occur only in the radial direction. The problem is then to compute the radial component

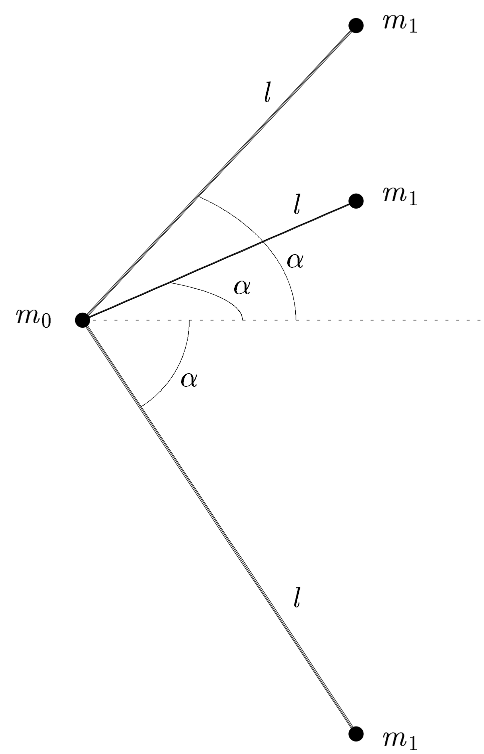

Figure 3: A quasi-rigid space-time swimmer. From the vertex, with mass m_0, extend three geodesic struts of proper length l. In a local Lorentz frame at the vertex each strut is tilted by the angle α from the axis of the swimmer. The three struts are equally spaced around this axis. The axis is pointing radially away from the central mass. The spatial Schwarzschild coordinates of the vertex are (r, θ, ϕ).

of the deformation field strength, which is a real-valued one-form as in the example of the plane and sphere.

The general form of the result can be anticipated, given the previous results. The displacement will be proportional to the square of ratio of the size of the object to the radius of curvature of the manifold. For Schwarzschild geometry the components of the Riemann curvature tensor are proportional to $GM/(c^2r^3)$, which may be thought of as the inverse of the square of an effective radius of curvature. So the displacement should be proportional to $l^2GM/(c^2r^3)$. In addition the displacement should be proportional to the change in length, and the change in separation angle. And there will be a factor that is homogeneous of degree zero in the masses. Detailed calculation confirms these expectations. For bodies that are small compared to the radius of curvature of space-time, the displacement is found to be

$$\Delta r = -\frac{3m_0m_1}{(m_0+3m_1)^2}\, l^2 \frac{GM}{c^2r^3}\, \sin\alpha\, \Delta l\, \Delta\alpha. \tag{74}$$

So it is indeed possible to swim in space-time. Translation in space can be accomplished merely by cylic changes in shape, without thrust or external forces.

The curvature of space-time is very slight, so the ability to swim in space-time is unlikely to lead to new propulsion devices. For a meter-sized object performing meter-sized deformations at the surface of the Earth, the displacement is of order 10^{-23} meters. Nevertheless, the effect is interesting as a matter of principle. You cannot lift yourself by pulling on your bootstraps, but you can lift yourself by kicking your heels.

8 Swimming in Acknowledgments

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